

# $K$ –structure of $\mathcal{U}(\mathfrak{g})$ for $\mathfrak{su}(n, 1)$ and $\mathfrak{so}(n, 1)$

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**Abstract.** Let  $G$  be the adjoint group of a real simple Lie algebra  $\mathfrak{g}_0$  equal either  $\mathfrak{su}(n, 1)$  or  $\mathfrak{so}(n, 1)$ ,  $K$  its maximal compact subgroup,  $\mathcal{U}(\mathfrak{g})$  the universal enveloping algebra of the complexification  $\mathfrak{g}$  of  $\mathfrak{g}_0$  and  $\mathcal{U}(\mathfrak{g})^K$  its subalgebra of  $K$ –invariant elements. By a result of F. Knopp [3]  $\mathcal{U}(\mathfrak{g})$  is free as a  $\mathcal{U}(\mathfrak{g})^K$ –module, so there exists a  $K$ –submodule  $E$  of  $\mathcal{U}(\mathfrak{g})$  such that the multiplication defines an isomorphism of  $K$ –modules  $\mathcal{U}(\mathfrak{g})^K \otimes E \rightarrow \mathcal{U}(\mathfrak{g})$ . We prove that  $E$  is equivalent to the regular representation of  $K$ , i.e. that the multiplicity of every  $\delta \in \hat{K}$  in  $E$  equals its dimension. As a consequence we get that for any finitedimensional complex  $K$ –module  $V$  the space  $(\mathcal{U}(\mathfrak{g}) \otimes V)^K$  of  $K$ –invariants is free  $\mathcal{U}(\mathfrak{g})^K$ –module of rank  $\dim V$ .

## 1 Introduction

Let  $\mathfrak{g}_0$  be a real simple Lie algebra of noncompact type. Denote by  $G$  its adjoint group and choose its maximal compact subgroup  $K$ . Let  $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$  be the corresponding Cartan decomposition. Let  $\mathfrak{g}$ ,  $\mathfrak{k}$  and  $\mathfrak{p}$  be the complexifications of  $\mathfrak{g}_0$ ,  $\mathfrak{k}_0$  and  $\mathfrak{p}_0$ , respectively. Denote by  $\mathcal{U}(\mathfrak{g})$  and  $\mathcal{U}(\mathfrak{k}) \subseteq \mathcal{U}(\mathfrak{g})$  the universal enveloping algebras of  $\mathfrak{g}$  and  $\mathfrak{k}$ . Furthermore, denote by  $S(\mathfrak{g})$  and  $S(\mathfrak{k}) \subseteq S(\mathfrak{g})$  the symmetric algebras over  $\mathfrak{g}$  and  $\mathfrak{k}$  and by  $\mathcal{P}(\mathfrak{g})$  and  $\mathcal{P}(\mathfrak{k})$  the polynomial algebras over  $\mathfrak{g}$  and  $\mathfrak{k}$ . Then  $\mathcal{P}(\mathfrak{g})$  and  $\mathcal{P}(\mathfrak{k})$  can be identified with the symmetric algebras  $S(\mathfrak{g}^*)$  and  $S(\mathfrak{k}^*)$  over dual spaces  $\mathfrak{g}^*$  and  $\mathfrak{k}^*$  of  $\mathfrak{g}$  and  $\mathfrak{k}$ . The Killing form  $B$  on  $\mathfrak{g}$  allows us to identify  $\mathfrak{g}$  with  $\mathfrak{g}^*$  and  $\mathfrak{k}$  with  $\mathfrak{k}^*$ . Thus the algebras  $\mathcal{P}(\mathfrak{g})$  and  $\mathcal{P}(\mathfrak{k})$  are identified with  $S(\mathfrak{g})$  and  $S(\mathfrak{k})$ . Considering polynomials as complex functions on  $\mathfrak{g}$  and  $\mathfrak{k}$ , the inclusion  $\mathcal{P}(\mathfrak{k}) \subseteq \mathcal{P}(\mathfrak{g})$  is obtained via the projection  $pr : \mathfrak{g} \rightarrow \mathfrak{k}$  along  $\mathfrak{p}$ .

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The adjoint action of the group  $G$  on  $\mathfrak{g}$  extends uniquely to the action by automorphisms on the algebras  $\mathcal{U}(\mathfrak{g})$ ,  $S(\mathfrak{g})$  and  $\mathcal{P}(\mathfrak{g})$ , and the subgroup  $K$  acts also by automorphisms on the algebras  $\mathcal{U}(\mathfrak{k})$ ,  $S(\mathfrak{k})$  and  $\mathcal{P}(\mathfrak{k})$ . Denote by superscript  $G$  (resp.  $K$ ) the subalgebras of  $G$ -invariants (resp.  $K$ -invariants). Then, of course,  $\mathcal{U}(\mathfrak{g})^G$  is the center  $\mathfrak{Z}(\mathfrak{g})$  of  $\mathcal{U}(\mathfrak{g})$  and  $\mathcal{U}(\mathfrak{k})^K$  is the center  $\mathfrak{Z}(\mathfrak{k})$  of  $\mathcal{U}(\mathfrak{k})$ . Obviously, the multiplication defines algebra homomorphisms

$$\mathfrak{Z}(\mathfrak{g}) \otimes \mathfrak{Z}(\mathfrak{k}) \rightarrow \mathcal{U}(\mathfrak{g})^K, \quad S(\mathfrak{g})^G \otimes S(\mathfrak{k})^K \rightarrow S(\mathfrak{g})^K, \quad \mathcal{P}(\mathfrak{g})^G \otimes \mathcal{P}(\mathfrak{k})^K \rightarrow \mathcal{P}(\mathfrak{g})^K.$$

In [3] F. Knopp has proved the following highly nontrivial results:

**Theorem 1.** (a)  $\mathfrak{Z}(\mathfrak{g}) \otimes \mathfrak{Z}(\mathfrak{k}) \rightarrow \mathcal{U}(\mathfrak{g})^K$  is an isomorphism onto the center of the algebra  $\mathcal{U}(\mathfrak{g})^K$ .

(b) The algebra  $\mathcal{U}(\mathfrak{g})^K$  is commutative (i.e.  $\mathcal{U}(\mathfrak{g})^K = \mathfrak{Z}(\mathfrak{g})\mathfrak{Z}(\mathfrak{k})$ ) if and only if  $\mathfrak{g}$  is either  $\mathfrak{su}(n, 1)$  or  $\mathfrak{so}(n, 1)$ . In these cases  $\mathcal{U}(\mathfrak{g})$  is free as a  $\mathcal{U}(\mathfrak{g})^K$ -module.

The symmetrization  $\mathcal{U}(\mathfrak{g}) \rightarrow S(\mathfrak{g}) \simeq \mathcal{P}(\mathfrak{g})$  is an isomorphism of vector spaces and of  $G$ -modules and (a) implies that the homomorphism

$$\mathcal{P}(\mathfrak{g})^G \otimes \mathcal{P}(\mathfrak{k})^K \rightarrow \mathcal{P}(\mathfrak{g})^K$$

is always injective and by (b) in the cases  $\mathfrak{g} = \mathfrak{su}(n, 1)$  and  $\mathfrak{g} = \mathfrak{so}(n, 1)$  this is an isomorphism; furthermore, the last sentence in (b) implies that in these two cases  $\mathcal{P}(\mathfrak{g})$  is free as a  $\mathcal{P}(\mathfrak{g})^K$ -module.

## 2 $K$ -harmonic polynomials and the structure of the $\mathcal{P}(\mathfrak{g})^K$ -module $\mathcal{P}(\mathfrak{g})$

Consider for a while a more general situation. Let  $V$  be a complex finite-dimensional vector space and let  $L$  be a closed subgroup of  $\mathrm{GL}(V)$  acting fully reducibly on  $V$ . Denote by  $S(V)$  and  $\mathcal{P}(V)$  the symmetric and the polynomial algebra over  $V$ . For  $x \in V$ , let  $\partial(x) : \mathcal{P}(V) \rightarrow \mathcal{P}(V)$  be the derivation in the direction  $x$ . The map  $\partial : V \rightarrow \mathrm{End}(\mathcal{P}(V))$  extends uniquely to an isomorphism  $\partial$  of the symmetric algebra  $S(V)$  onto the algebra  $\mathcal{D}(V)$  of

linear differential operators on  $\mathcal{P}(V)$  with constant coefficients. Now, one defines the bilinear form  $\langle \cdot, \cdot \rangle$  on  $S(V) \times \mathcal{P}(V)$  by

$$\langle u, f \rangle = [\partial(u)f](0), \quad u \in S(V), \quad f \in \mathcal{P}(V).$$

This is a pairing, i.e. nondegenerate in each variable. Now, consider the subalgebras of  $L$ -invariants  $S(V)^L$  and  $\mathcal{P}(V)^L$  and their maximal ideals (of codimension 1)

$$S_+(V)^L = \bigoplus_{k>0} S^k(V)^L, \quad \mathcal{P}_+(V)^L = \bigoplus_{k>0} \mathcal{P}^k(V)^L = \{f \in \mathcal{P}(V)^L; f(0) = 0\}.$$

Define the (graded) space of so called  $L$ -harmonic polynomials on  $V$  :

$$\mathcal{H}_L(V) = \{f \in \mathcal{P}(V); \partial(u)f = 0 \quad \forall u \in S_+(V)^L\}.$$

As noticed in [4] and [5] the obvious equality

$$\langle uv, f \rangle = \langle u, \partial(v)f \rangle, \quad u, v \in S(V), \quad f \in \mathcal{P}(V),$$

implies easily that

$$\mathcal{H}_L(V) = \{f \in \mathcal{P}(V); \langle u, f \rangle = 0 \quad \forall u \in S(V)S_+(V)^L\}.$$

Part of the Helgason's results in [2] (see also Propositions 3 and 4 in [4]) can be stated as follows:

**Proposition 1.** *Suppose that the group  $L$  is connected and that there exists an  $L$ -invariant symmetric bilinear form  $B : V \times V \longrightarrow \mathbb{C}$  and a real form  $V_0$  of  $V$  such that the restriction of  $B$  to  $V_0 \times V_0$  is a scalar product and that the group  $L$  is the complexification of its subgroup  $L_0 = \{g \in L; gV_0 = V_0\}$ . Then*

$$\mathcal{P}(V) = \mathcal{P}(V)\mathcal{P}_+(V)^L \oplus \mathcal{H}_L(V).$$

Note that the conditions on the pair  $(L, V)$  in Proposition 1 are obviously satisfied for the action of the complexification  $K^{\mathbb{C}}$  of the group  $K$  on  $\mathfrak{g}$ , especially in the cases  $\mathfrak{g}_0 = \mathfrak{su}(n, 1)$  and  $\mathfrak{g}_0 = \mathfrak{so}(n, 1)$ .

Consider any subgroup  $L \subseteq \mathrm{GL}(V)$  acting fully reducibly on a finitedimensional complex vector space  $V$ . If  $N$  is any graded subspace of  $\mathcal{P}(V)$  such that

$$\mathcal{P}(V) = \mathcal{P}(V)\mathcal{P}_+(V)^L \oplus N \tag{1}$$

then it is easy to see (Proposition 1 in [4]) that the multiplication defines a surjective map

$$\mathcal{P}(V)^L \otimes N \longrightarrow \mathcal{P}(V). \quad (2)$$

Kostant's Lemma 1 in [4] can be stated as follows:

**Proposition 2.** *The following properties are mutually equivalent:*

- (a) *For every  $N$ , such that (1) holds true, the map (2) is also injective, i.e. an isomorphism.*
- (b) *For some  $N$ , such that (1) holds true, the map (2) is injective.*
- (c)  *$\mathcal{P}(V)$  is free as a  $\mathcal{P}(V)^L$ -module.*

Thus, by the last sentence in (b) of Theorem 1 we get from Propositions 1 and 2:

**Theorem 2.** *For  $\mathfrak{g} = \mathfrak{su}(n, 1)$  and for  $\mathfrak{g} = \mathfrak{so}(n, 1)$  we have:*

- (a)  $\mathcal{P}(\mathfrak{g}) = \mathcal{P}(\mathfrak{g})\mathcal{P}_+(\mathfrak{g})^K \oplus \mathcal{H}_K(\mathfrak{g})$ .
- (b) *The multiplication defines an isomorphism  $\mathcal{P}(\mathfrak{g})^K \otimes \mathcal{H}_K(\mathfrak{g}) \simeq \mathcal{P}(\mathfrak{g})$ .*

### 3 The $K$ -module of $K$ -harmonic polynomials

Let  $\mathcal{N}$  be the zero set in  $\mathfrak{g}$  of the ideal  $\mathcal{P}(\mathfrak{g})\mathcal{P}_+(\mathfrak{g})^K$  generated by  $\mathcal{P}_+(\mathfrak{g})^K$  in  $\mathcal{P}(\mathfrak{g})$  :

$$\mathcal{N} = \{x \in \mathfrak{g}; f(x) = 0 \ \forall f \in \mathcal{P}(\mathfrak{g})\mathcal{P}_+(\mathfrak{g})^K\} = \{x \in \mathfrak{g}; f(x) = 0 \ \forall f \in \mathcal{P}_+(\mathfrak{g})^K\}.$$

By Proposition 16 in [4] the zero set

$$\mathcal{N}_G = \{x \in \mathfrak{g}; f(x) = 0 \ \forall f \in \mathcal{P}_+(\mathfrak{g})^G\}$$

is exactly the set of all nilpotent elements in the Lie algebra  $\mathfrak{g}$ . Analogously

$$\mathcal{N}_K = \{x \in \mathfrak{k}; f(x) = 0 \ \forall f \in \mathcal{P}_+(\mathfrak{k})^K\}$$

is the set of all nilpotent elements in the reductive Lie algebra  $\mathfrak{k}$ . Now,  $\mathcal{P}(\mathfrak{g})^K = \mathcal{P}(\mathfrak{g})^G \otimes \mathcal{P}(\mathfrak{k})^K$  by the Knopp's theorem, so we get

**Proposition 3.**  $\mathcal{N}$  is the set of all nilpotent elements in  $\mathfrak{g}$  whose projection to  $\mathfrak{k}$  along  $\mathfrak{p}$  is nilpotent in the reductive Lie algebra  $\mathfrak{k}$  :

$$\mathcal{N} = \{x \in \mathfrak{g}; x \in \mathcal{N}_G, \text{ pr } x \in \mathcal{N}_K\}.$$

We call the elements of  $\mathcal{N}$   $K$ -nilpotent elements in  $\mathfrak{g}$ .

By the Harish–Chandra isomorphism and by the Chevalley’s theorem on Weyl group invariants we know that the algebra  $\mathcal{P}(\mathfrak{g})^G$  is generated by  $\ell = \text{rank } \mathfrak{g}$  homogeneous algebraically independent  $G$ -invariant polynomials  $f_1, \dots, f_\ell$  and the algebra  $\mathcal{P}(\mathfrak{k})^K$  is generated by  $k = \text{rank } \mathfrak{k}$  homogeneous algebraically independent  $K$ -invariant polynomials  $\varphi_1, \dots, \varphi_k$ . Since in the cases  $\mathfrak{g}_0 = \mathfrak{su}(n, 1)$  and  $\mathfrak{g}_0 = \mathfrak{so}(n, 1)$

$$\mathcal{P}(\mathfrak{g})^K = \mathcal{P}(\mathfrak{g})^G \mathcal{P}(\mathfrak{k})^K \simeq \mathcal{P}(\mathfrak{g})^G \otimes \mathcal{P}(\mathfrak{k})^K,$$

the algebra  $\mathcal{P}(\mathfrak{g})^K$  is generated by  $\ell + k$  homogeneous algebraically independent polynomials  $f_1, \dots, f_\ell, \varphi_1, \dots, \varphi_k$ . Thus,

$$\mathcal{N} = \{x \in \mathfrak{g}; f_1(x) = \dots = f_\ell(x) = \varphi_1(x) = \dots = \varphi_k(x) = 0\},$$

so the set  $\mathcal{N}$  is a Zariski closed subset of  $\mathfrak{g}$  of dimension

$$\dim \mathcal{N} = \dim \mathfrak{g} - \ell - k.$$

More generally, for any  $(\xi, \eta) = (\xi_1, \dots, \xi_\ell, \eta_1, \dots, \eta_k) \in \mathbb{C}^{\ell+k}$  we define a  $K^\mathbb{C}$ -stable Zariski closed subset  $\mathcal{N}(\xi, \eta)$  of  $\mathfrak{g}$  :

$$\mathcal{N}(\xi, \eta) = \{x \in \mathfrak{g}; f_j(x) = \xi_j, j = 1, \dots, \ell, \varphi_i(x) = \eta_i, i = 1, \dots, k\}.$$

Obviously,

$$\dim \mathcal{N}(\xi, \eta) = \dim \mathfrak{g} - \ell - k, \quad (\xi, \eta) \in \mathbb{C}^{\ell+k}.$$

As in [4] and [5] we conclude from Theorem 2(a) :

**Proposition 4.** *The restriction of polynomials in  $\mathcal{P}(\mathfrak{g})$  to the set  $\mathcal{N}(\xi, \eta)$  induces an isomorphism of  $K$ -modules*

$$\mathcal{H}_K(\mathfrak{g}) \simeq \mathcal{P}(\mathcal{N}(\xi, \eta)) = \mathcal{R}(\mathcal{N}(\xi, \eta)), \quad (\xi, \eta) \in \mathbb{C}^{k+\ell}.$$

Here for any subset  $S \subseteq \mathfrak{g}$  we set

$$\mathcal{P}(S) = \{f|S; f \in \mathcal{P}(\mathfrak{g})\}$$

and for any algebraic variety  $S$   $\mathcal{R}(S)$  denotes the algebra of regular functions on  $S$ .

The dimensions and the ranks  $\ell = \text{rank } \mathfrak{g}$  and  $k = \text{rank } \mathfrak{k}$  in our cases are the following:

$\mathfrak{g}$	$\dim \mathfrak{g}$	$\dim \mathfrak{k}$	$\ell$	$k$
$\mathfrak{su}(n, 1)$	$n^2 + 2n$	$n^2$	$n$	$n$
$\mathfrak{so}(2n, 1)$	$2n^2 + n$	$2n^2 - n$	$n$	$n$
$\mathfrak{so}(2n + 1, 1)$	$2n^2 + 3n + 1$	$2n^2 + n$	$n + 1$	$n$

So we see that in each case

$$\dim \mathcal{N}(\xi, \eta) = \dim \mathfrak{k} = \dim K^{\mathbb{C}}, \quad (\xi, \eta) \in \mathbb{C}^{\ell+k}. \quad (3)$$

**Remark:** By the exercise 13) in §13 in [1] (p. 268) we can choose the following generators  $f_i, \varphi_j$  of  $\mathcal{P}(\mathfrak{g})^K$ :

(a) For  $\mathfrak{g}_0 = \mathfrak{su}(n, 1)$

$$f_i(x) = \text{Tr } x^{i+1}, \quad 1 \leq i \leq n, \quad \varphi_j(x) = \text{Tr } (pr x)^j, \quad 1 \leq j \leq n.$$

(b) For  $\mathfrak{g}_0 = \mathfrak{so}(2n, 1)$

$$f_i(x) = \text{Tr } x^{2i}, \quad 1 \leq i \leq n, \quad \varphi_j(x) = \text{Tr } (pr x)^{2j}, \quad 1 \leq j \leq n-1, \\ \varphi_n(x)^2 = (-1)^n \det (pr x).$$

(c) For  $\mathfrak{g}_0 = \mathfrak{so}(2n + 1, 1)$

$$f_i(x) = \text{Tr } x^{2i}, \quad 1 \leq i \leq n, \quad f_{n+1}(x)^2 = (-1)^{n+1} \det x, \\ \varphi_j(x) = \text{Tr } (pr x)^{2j}, \quad 1 \leq j \leq n.$$

Consider the action of the complex group  $K^{\mathbb{C}}$  on  $\mathfrak{g}$ . For  $x \in \mathfrak{g}$  denote by  $\mathcal{O}_x$  its  $K^{\mathbb{C}}$ -orbit. Then of course

$$\dim \mathcal{O}_x = \dim K^{\mathbb{C}} / K_x^{\mathbb{C}} = \dim K^{\mathbb{C}} - \dim K_x^{\mathbb{C}}, \quad (4)$$

where  $K_x^{\mathbb{C}}$  denotes the stabilizer of the point  $x$  in the group  $K^{\mathbb{C}}$ . So, if  $K_x^{\mathbb{C}}$  is trivial

$$\dim \mathcal{O}_x = \dim K^{\mathbb{C}} = \dim \mathcal{N}(\xi, \eta). \quad (5)$$

**Lemma 1.** *There exists  $x \in \mathfrak{g}$  such that the stabilizer  $K_x^{\mathbb{C}}$  is trivial. In this case let  $(\xi, \eta) = (f_1(x), \dots, f_\ell(x), \varphi_1(x), \dots, \varphi_k(x))$ . The orbit  $\mathcal{O}_x$  is open in  $\mathcal{N}(\xi, \eta)$ .*

We prove this Lemma in Section 4.

Let  $x \in \mathfrak{g}$  be as in Lemma 1, i.e. such that its stabilizer in  $K^{\mathbb{C}}$  is trivial. Set

$$(\xi, \eta) = (f_1(x), \dots, f_\ell(x), \varphi_1(x), \dots, \varphi_k(x)) \in \mathbb{C}^{\ell+k}.$$

We know that  $\dim \mathcal{O}_x = \dim \mathcal{N}(\xi, \eta)$ , so the  $K^{\mathbb{C}}$ -orbit  $\mathcal{O}_x$  is open in  $\mathcal{N}(\xi, \eta)$ . Thus, the restriction to  $\mathcal{O}_x$  is an isomorphism of  $\mathcal{P}(\mathcal{N}(\xi, \eta)) = \mathcal{R}(\mathcal{N}(\xi, \eta))$  onto  $\mathcal{P}(\mathcal{O}_x)$ . Now, if the algebraic variety  $\mathcal{N}(\xi, \eta)$  would be irreducible and if we would have

$$\dim \mathcal{N}(\xi, \eta) \setminus \mathcal{O}_x \leq \dim \mathcal{N}(\xi, \eta) - 2, \quad (6)$$

(this holds true in the settings of [4] and [5] since the dimensions of all the orbits have the same parity) we could conclude by a theorem from algebraic geometry that  $\mathcal{P}(\mathcal{O}_x) = \mathcal{R}(\mathcal{O}_x) \simeq \mathcal{R}(K^{\mathbb{C}})$  as  $K^{\mathbb{C}}$ -modules and by the Frobenius reciprocity we could get that the multiplicity  $m(\delta)$  of any irreducible finitedimensional representation  $\delta$  of  $K^{\mathbb{C}}$  in the  $K^{\mathbb{C}}$ -module  $\mathcal{H}_K(\mathfrak{g}) \simeq \mathcal{R}(\mathcal{O}_x)$  equals its dimension  $d(\delta)$ . Unfortunately, (6) is not true. In fact, in the case  $\mathfrak{g} = \mathfrak{su}(n, 1)$  the algebraic set  $\mathcal{N} = \mathcal{N}(0, 0)$  is even not irreducible – there exist two open orbits in  $\mathcal{N}$ , and in the complement of these two orbits there exist orbits of dimension  $\dim \mathcal{N} - 1$ . In the case  $\mathfrak{g} = \mathfrak{so}(n, 1)$ ,  $n \geq 3$ , there also exist  $K^{\mathbb{C}}$ -orbits in  $\mathcal{N}(\xi, \eta)$  of dimension  $\mathcal{N}(\xi, \eta) - 1$ .

So, we get only the inclusion of  $K$ -modules  $\mathcal{H}_K(\mathfrak{g}) \hookrightarrow \mathcal{R}(K^{\mathbb{C}})$  and we may conclude only that

$$m(\delta) \leq d(\delta) \quad (7)$$

for every irreducible finitedimensional representation  $\delta$  of  $K$ . In fact, the equality holds true although we do not know *a priori* that  $\mathcal{P}(\mathcal{O}_x) = \mathcal{R}(\mathcal{O}_x)$ ; it comes out *a posteriori*.

**Theorem 3.** *The multiplicity of every irreducible finitedimensional representation  $\delta$  of the compact group  $K$  in the  $K$ -module  $\mathcal{H}_K(\mathfrak{g})$  of  $K$ -harmonic polynomials on  $\mathfrak{g}$  is equal to its dimension  $d(\delta)$ .*

To prove Theorem 3 we use the compact form  $K$  of the complex group  $K^{\mathbb{C}}$ . Denote by  $\mathcal{P}(Kx)$  the restriction of the polynomial algebra  $\mathcal{P}(\mathfrak{g})$  to the

$K$ -orbit  $Kx$ . Note that the fact that  $K^{\mathbb{C}}$  is the complexification of  $K$  easily implies that the restriction  $\mathcal{O}_x \rightarrow Kx$  induces an isomorphism of  $K$ -modules  $\mathcal{P}(\mathcal{O}_x)$  onto  $\mathcal{P}(Kx)$ . Thus, as a  $K$ -module we have

$$\mathcal{P}(Kx) = \bigoplus_{\delta \in \hat{K}} m(\delta) \delta. \quad (8)$$

The subalgebra  $\mathcal{P}(Kx)$  of the algebra  $C(Kx)$  of all complex continuous functions on the compact space  $Kx$  evidently distinguishes the points of  $Kx$ . Furthermore, this subalgebra is closed under complex conjugation. This is implied by the fact that the set  $Kx$  is contained in a real form of the complex vector space  $\mathfrak{g}$ . This follows from the fact that the compact group  $K$  is contained in a maximal compact subgroup  $U$  of the complex group  $G^{\mathbb{C}} = \text{Int}(\mathfrak{g})$  and the Lie algebra  $\mathfrak{u}$  of  $U$  is a real form of  $\mathfrak{g}$ . Finally, the algebra  $\mathcal{P}(Kx)$  obviously contains constants. Thus, by the Stone–Weierstrass theorem the subalgebra  $\mathcal{P}(Kx)$  is uniformly dense in  $C(Kx)$ . Now, the Peter–Weyl theorem implies that  $m(\delta) = d(\delta)$  for all  $\delta \in \hat{K}$ . This proves Theorem 3.

The symmetrization  $U(\mathfrak{g}) \rightarrow S(\mathfrak{g}) \simeq \mathcal{P}(\mathfrak{g})$  is an  $K$ -module isomorphism. Let  $H_K$  be the inverse image of  $\mathcal{H}_K(\mathfrak{g})$  in  $U(\mathfrak{g})$ . The immediate consequence of Theorems 2 and 3 is

**Theorem 4.** *The multiplication induces an isomorphism of  $K$ -modules  $U(\mathfrak{g})^K \otimes H_K \simeq U(\mathfrak{g})$ . The multiplicity of every  $\delta \in \hat{K}$  in the  $K$ -module  $H_K$  is equal to its dimension  $d(\delta)$ .*

**Corollary 1.** *Let  $V$  be a finitedimensional  $K$ -module. Then the space of  $K$ -invariants  $(U(\mathfrak{g}) \otimes V)^K$  is a free  $U(\mathfrak{g})^K$ -module of finite rank  $\dim V$ .*

By Theorem 4 we have

$$(U(\mathfrak{g}) \otimes V)^K \simeq (U(\mathfrak{g})^K \otimes H_K \otimes V)^K = U(\mathfrak{g})^K \otimes (H_K \otimes V)^K.$$

Thus,  $U(\mathfrak{g})$  is a free  $U(\mathfrak{g})^K$ -module of rank  $\dim (H_K \otimes V)^K$ . Now, let  $n(\varepsilon)$  be the multiplicity of  $\varepsilon \in \hat{K}$  in  $V$ . Then

$$(H_K \otimes V)^K \simeq \left( \left( \bigoplus_{\delta \in \hat{K}} d(\delta) \delta \right) \otimes \left( \bigoplus_{\varepsilon \in \hat{K}} n(\varepsilon) \varepsilon \right) \right)^K = \bigoplus_{\delta, \varepsilon \in \hat{K}} d(\delta) n(\varepsilon) (\delta \otimes \varepsilon)^K,$$

so

$$\dim (H_K \otimes V)^K = \sum_{\delta, \varepsilon \in \hat{K}} d(\delta) n(\varepsilon) \dim (\delta \otimes \varepsilon)^K.$$



By the Schur's lemma  $\dim(\delta \otimes \varepsilon)^K$  is 1 if  $\delta$  and  $\varepsilon$  are contragredient to each other and 0 otherwise. Since the dimensions of contragredient representations are equal, we get

$$\dim(H_K \otimes V)^K = \sum_{\delta \in \hat{K}} n(\delta) d(\delta) = \dim V.$$

## 4 Proof of Lemma 1

(1)  $\mathfrak{g}_0 = \mathfrak{su}(n, 1)$ . We realize this Lie algebra as

$$\mathfrak{g}_0 = \{A \in \mathfrak{sl}(n+1, \mathbb{C}); A^* = -\Gamma A \Gamma\},$$

where  $\Gamma = \text{diag}(1, \dots, 1, -1)$ . Then  $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$  and  $K^{\mathbb{C}} = \tilde{K}^{\mathbb{C}}/Z$ , where  $Z = \{\text{diag}(\alpha, \dots, \alpha); \alpha^{n+1} = 1\}$  is the center of  $\text{SL}(n+1, \mathbb{C})$  and

$$\tilde{K}^{\mathbb{C}} = \left\{ \begin{bmatrix} B & 0 \\ 0 & (\det B)^{-1} \end{bmatrix}; B \in \text{GL}(n, \mathbb{C}) \right\}.$$

Now, we can take for  $x$  the elementary  $(n+1) \times (n+1)$  Jordan block:

$$x = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

The centralizer  $M_x$  of  $x$  in the algebra of all  $(n+1) \times (n+1)$  matrices consists of all polynomials in  $x$ , i.e.

$$M_x = \left\{ \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & \alpha_n \\ 0 & \alpha_0 & \alpha_1 & \cdots & \alpha_{n-2} & \alpha_{n-1} \\ 0 & 0 & \alpha_0 & \cdots & \alpha_{n-3} & \alpha_{n-2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_0 & \alpha_1 \\ 0 & 0 & 0 & \cdots & 0 & \alpha_0 \end{bmatrix}; \alpha_0, \alpha_1, \dots, \alpha_n \in \mathbb{C} \right\}.$$

So, we conclude that the centralizer of  $x$  in  $\tilde{K}^{\mathbb{C}}$  is precisely the center  $Z$  of  $\text{SL}(n+1, \mathbb{C})$ , thus the stabilizer of  $x$  in  $K^{\mathbb{C}}$  is trivial.

(2)  $\mathfrak{g}_0 = \mathfrak{so}(2n+1, 1)$ . We choose the following realizations:

$$\mathfrak{g} = \mathfrak{so}(2n+2, \mathbb{C}) = \{A \in \mathfrak{gl}(2n+2, \mathbb{C}); A^t = -\Gamma A \Gamma\},$$

$$\mathfrak{k} = \left\{ \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}; B \in \mathfrak{gl}(2n+1, \mathbb{C}), B^t = -\Gamma_0 B \Gamma_0 \right\}.$$

Here the superscript  $t$  denotes the matrix transpose and

$$\Gamma_0 = \begin{bmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} \Gamma_0 & 0 \\ 0 & 1 \end{bmatrix},$$

$I_n$  being the  $n$  by  $n$  identity matrix. Denoting as usual the space of all  $n \times m$  complex matrices by  $M_{n,m}(\mathbb{C})$  and  $M_n(\mathbb{C}) = M_{n,n}(\mathbb{C})$ , we have

$$\mathfrak{k} = \left\{ \begin{bmatrix} A & B & a & 0 \\ C & -A^t & b & 0 \\ -b^t & -a^t & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; A, B, C \in M_n(\mathbb{C}), B^t = -B, C^t = -C, a, b \in M_{n,1}(\mathbb{C}) \right\}$$

and

$$\mathfrak{g} = \left\{ X + \begin{bmatrix} 0 & 0 & 0 & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & \alpha \\ -d^t & -c^t & -\alpha & 0 \end{bmatrix}; X \in \mathfrak{k}, c, d \in M_{n,1}(\mathbb{C}), \alpha \in \mathbb{C} \right\}.$$

Furthermore,

$$K^{\mathbb{C}} = \left\{ \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}; A \in \mathrm{SL}(2n+1, \mathbb{C}), A^t \Gamma_0 A = \Gamma_0 \right\}.$$

Let  $J$  denote the elementary  $n$  by  $n$  Jordan block and let  $e_j \in M_{n,1}(\mathbb{C})$  be the column matrix with 1 in the  $j$ -th row and zeros elsewhere. Set

$$x = \begin{bmatrix} J & 0 & e_n & 0 \\ 0 & -J^t & 0 & e_1 \\ 0 & -e_n^t & 0 & 0 \\ -e_1^t & 0 & 0 & 0 \end{bmatrix}.$$

This element of  $\mathfrak{g}$  is an invertible matrix which is up to the change of some signs (to be precise, on the places  $1, n+1, n+2, \dots, 2n$ ) the matrix of the following cyclic permutation of the set of indices  $\{1, 2, \dots, 2n+2\}$ :

$$1 \rightarrow 2 \rightarrow \dots \rightarrow n-1 \rightarrow n \rightarrow 2n+1 \rightarrow 2n+1 \rightarrow n+1 \rightarrow n+2 \rightarrow \dots \rightarrow 2n \rightarrow 1.$$

Thus, we conclude that the stabilizer (i.e. the centralizer) of  $x$  in  $K^{\mathbb{C}}$  is trivial.

(3)  $\mathfrak{g} = \mathfrak{so}(2n, 1)$ . We choose the following realizations

$$\mathfrak{g} = \mathfrak{so}(2n + 1, \mathbb{C}) = \{A \in \mathfrak{gl}(2n + 1, \mathbb{C}); A^t = -\Gamma A \Gamma\},$$

$$\mathfrak{k} = \left\{ \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}; B \in \mathfrak{gl}(2n, \mathbb{C}), B^t = -\Gamma_0 B \Gamma_0 \right\},$$

$$\Gamma_0 = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} \Gamma_0 & 0 \\ 0 & 1 \end{bmatrix},$$

Then

$$\mathfrak{k} = \left\{ \begin{bmatrix} A & B & 0 \\ C & -A^t & 0 \\ 0 & 0 & 0 \end{bmatrix}; A, B, C \in M_n(\mathbb{C}), B^t = -B, C^t = -C \right\}$$

and

$$\mathfrak{g} = \left\{ X + \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ -b^t & -a^t & 0 \end{bmatrix}; X \in \mathfrak{k}, a, b \in M_{n,1}(\mathbb{C}) \right\}.$$

As in (2) let  $J$  denote the elementary  $n$  by  $n$  Jordan block and let

$$\Delta = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & -1 & 0 \end{bmatrix} \in M_n(\mathbb{C}).$$

The matrix

$$x_0 = \begin{bmatrix} J & \Delta & 0 \\ 0 & -J^t & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

is a representative of the  $K^\mathbb{C}$ -orbit of all principal nilpotent elements of  $\mathfrak{k}$ . By the Kostant's results in [3] the stabilizer  $K_{x_0}^\mathbb{C}$  of  $x_0$  in  $K^\mathbb{C}$  is an  $n$ -dimensional connected simply connected unipotent subgroup whose Lie algebra is the centralizer  $\mathfrak{k}_{x_0}$  of  $x_0$  in  $\mathfrak{k}$ .

(3a) Suppose first that  $n$  is odd,  $n = 2k + 1$ . By solving a system of linear equations one finds that  $\mathfrak{k}_{x_0}$  consists of all matrices of the form

$$\begin{bmatrix} A & B & 0 \\ 0 & -A^t & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (9)$$

where  $B$  is  $n$  by  $n$  antisymmetric matrix such that for some  $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{C}$  its first row is

$$\begin{bmatrix} 0 & \alpha_1 & 0 & \alpha_2 & 0 & \cdots & 0 & \alpha_k & \alpha_{k+1} \end{bmatrix},$$

its last column is

$$\begin{bmatrix} \alpha_{k+1} & \alpha_{k+2} & 0 & \alpha_{k+3} & 0 \cdots & 0 & \alpha_{2k+1} & 0 \end{bmatrix}^t,$$

the inner entries of  $B$  are either 0, or  $\pm\alpha_j$ ,  $2 \leq j \leq k$ , or  $\pm 2\alpha_j$ ,  $k+2 \leq j \leq 2k$ , and  $A$  is a strictly upper triangular  $n$  by  $n$  matrix whose first row is

$$\begin{bmatrix} 0 & \alpha_{2k+1} & 0 & \alpha_{2k} & 0 & \cdots & 0 & \alpha_{k+2} & -\alpha_{k+1} \end{bmatrix}$$

and every parallel with the main diagonal is constant (i.e.  $A$  is a polynomial in  $J$ ). E.g. for  $n = 7$  ( $k = 3$ )

$$A = \begin{bmatrix} 0 & \alpha_7 & 0 & \alpha_6 & 0 & \alpha_5 & -\alpha_4 \\ 0 & 0 & \alpha_7 & 0 & \alpha_6 & 0 & \alpha_5 \\ 0 & 0 & 0 & \alpha_7 & 0 & \alpha_6 & 0 \\ 0 & 0 & 0 & 0 & \alpha_7 & 0 & \alpha_6 \\ 0 & 0 & 0 & 0 & 0 & \alpha_7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & \alpha_1 & 0 & \alpha_2 & 0 & \alpha_3 & \alpha_4 \\ -\alpha_1 & 0 & -\alpha_2 & 0 & -\alpha_3 & 0 & \alpha_5 \\ 0 & \alpha_2 & 0 & \alpha_3 & 0 & -2\alpha_5 & 0 \\ -\alpha_2 & 0 & -\alpha_3 & 0 & 2\alpha_5 & 0 & \alpha_6 \\ 0 & \alpha_3 & 0 & -2\alpha_5 & 0 & -2\alpha_6 & 0 \\ -\alpha_3 & 0 & 2\alpha_5 & 0 & 2\alpha_6 & 0 & \alpha_7 \\ -\alpha_4 & -\alpha_5 & 0 & -\alpha_6 & 0 & -\alpha_7 & 0 \end{bmatrix}.$$

(3b) Consider now the case of  $n$  even,  $n = 2k$ . As in (3a) one finds that  $\mathfrak{k}_{x_0}$  consists of all matrices of the form (9) where  $B$  is  $n$  by  $n$  antisymmetric matrix whose first row is

$$\begin{bmatrix} 0 & \alpha_1 & 0 & \alpha_2 & 0 & \cdots & 0 & \alpha_k \end{bmatrix},$$

its last column is

$$\begin{bmatrix} \alpha_k & 0 & \alpha_{k+2} & 0 & \alpha_{k+3} & 0 & \cdots & 0 & \alpha_{2k} & 0 \end{bmatrix}^t,$$

the inner entries of its antidiagonal are  $\pm\alpha_{k+1}$ , all the other inner entries are either 0, or  $\pm\alpha_j$ ,  $2 \leq j \leq k-1$ , or  $\pm 2\alpha_j$ ,  $k+2 \leq j \leq 2k-1$ , and  $A$  is the strictly upper triangular  $n$  by  $n$  matrix whose first row is

$$\begin{bmatrix} 0 & \alpha_{2k} & 0 & \alpha_{2k-1} & 0 & \cdots & 0 & \alpha_{k+2} & 0 & \alpha_{k+1} - \alpha_k \end{bmatrix}$$

and every paralel with the main diagonal is constant. E.g. for  $n = 6$  ( $k = 3$ )

$$A = \begin{bmatrix} 0 & \alpha_6 & 0 & \alpha_5 & 0 & \alpha_4 - \alpha_3 \\ 0 & 0 & \alpha_6 & 0 & \alpha_5 & 0 \\ 0 & 0 & 0 & \alpha_6 & 0 & \alpha_5 \\ 0 & 0 & 0 & 0 & \alpha_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & \alpha_1 & 0 & \alpha_2 & 0 & \alpha_3 \\ -\alpha_1 & 0 & -\alpha_2 & 0 & -\alpha_4 & 0 \\ 0 & \alpha_2 & 0 & \alpha_4 & 0 & \alpha_5 \\ -\alpha_2 & 0 & -\alpha_4 & 0 & -2\alpha_5 & 0 \\ 0 & \alpha_4 & 0 & 2\alpha_5 & 0 & \alpha_6 \\ -\alpha_3 & 0 & -\alpha_5 & 0 & -\alpha_6 & 0 \end{bmatrix}.$$

Now, since  $\mathfrak{p}$  is  $K^\mathbb{C}$ -stable, for any  $y \in \mathfrak{p}$  the stabilizer (resp. the centralizer) of  $x = x_0 + y$  in  $K^\mathbb{C}$  (resp.  $\mathfrak{k}$ ) is the stabilizer (resp. the centralizer) of  $y$  in  $K_{x_0}^\mathbb{C}$  (resp.  $\mathfrak{k}_{x_0}$ ). Let us compute the centralizer of

$$y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & e_1 \\ -e_1^t & 0 & 0 \end{bmatrix} \in \mathfrak{p}^\mathbb{C}$$

in  $\mathfrak{k}_{x_0}$ . An element (9) of  $\mathfrak{k}_{x_0}$  centralizes  $y$  if and only if

$$Be_1 = 0 \quad \text{and} \quad A^t e_1 = 0.$$

Now, in the case (3a) we have

$$Be_1 = \begin{bmatrix} 0 & -\alpha_1 & 0 & -\alpha_2 & 0 & \cdots & -\alpha_k & -\alpha_{k+1} \end{bmatrix}^t,$$

$$A^t e_1 = \begin{bmatrix} 0 & \alpha_{2k+1} & 0 & \alpha_{2k} & 0 & \cdots & 0 & \alpha_{k+2} & -\alpha_{k+1} \end{bmatrix}^t,$$

and in the case (3b)

$$Be_1 = \begin{bmatrix} 0 & -\alpha_1 & 0 & -\alpha_2 & 0 & \cdots & 0 & -\alpha_k \end{bmatrix}^t,$$

$$A^t e_1 = \begin{bmatrix} 0 & \alpha_{2k} & 0 & \alpha_{2k-1} & 0 & \vdots & 0 & \alpha_{k+2} & 0 & \alpha_{k+1} - \alpha_k \end{bmatrix}^t.$$

In both cases we conclude that (5) is in the centralizer of  $y$  in  $\mathfrak{k}_{x_0}$  if and only if  $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ , i.e. if and only if  $A = B = 0$ . Thus,

$$x = x_0 + y = \begin{bmatrix} J & \Delta & 0 \\ 0 & -J^t & e_1 \\ -e_1^t & 0 & 0 \end{bmatrix}$$

is an element of  $\mathfrak{g}$  whose stabilizer in  $K^{\mathbb{C}}$  is trivial. This completes the proof of Lemma 1.

## References

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